Spring 2001

ABSTRACTING ARISTOTLE'S PHILOSOPHY OF MATHEMATICS

John J. Cleary
NUI Maynooth, babich@fordham.edu

JOHN J. CLEARY

ABSTRACTING ARISTOTLE'S PHILOSOPHY OF MATHEMATICS

In the history of science perhaps the most influential Aristotelian division was that between mathematics and physics. From our modern perspective this seems like an unfortunate deviation from the Platonic unification of the two disciplines, which guided Kepler and Galileo towards the modern scientific revolution. By contrast, Aristotle's sharp distinction between the disciplines seems to have led to a barren scholasticism in physics, together with an arid instrumentalism in Ptolemaic astronomy. On the positive side, however, astronomy was liberated from commonsense realism for the conceptual experiments of Aristarchus of Samos, whose heliocentric hypothesis was not adopted by later astronomers because it departed so much from the ancient cosmological consensus. It was only in the time of Newton that convincing physical arguments were able to overcome the legitimate objections against heliocentrism, which had looked like a mathematical hypothesis with no physical meaning.

Thus from the perspective of the history of science, as well as from that of Aristotelian scholarship, it is important to examine the details of Aristotle's philosophy of mathematics with particular attention to its relationship with the physical world, as reflected in the so-called 'mixed' sciences of astronomy, optics and mechanics. Furthermore, we face a deep hermeneutical problem in trying to understand Aristotle's philosophy of mathematics without drawing false parallels with modern views that were developed in response to the foundational crisis at the end of the 19th century. On the one hand, it is an inescapable fact about our mode of understanding that we cannot jump over our own shadow, as it were; so that we cannot avoid asking whether Aristotle was a platonist, or an intuitionist, or a logicist, or a formalist, or some kind of quasi-empiricist. When pursued in this way, the attempt to grapple with Aristotle's philosophy of mathematics is reduced to asking how well his view matches one of the standard modern views that were developed within an entirely different problem-situation in the history of philosophy. But, on the other hand, one wonders whether it is even possible to recover the original problem-situation in which Aristotle's views about mathematics were developed.

B. E. Babich (ed..), Philosophy of Science, Van Gogh's Eyes, and God: Hermeneutic Essays in Honor of Patrick A. Heelan, S.J., 163–176.

THE ROLE OF MATHEMATICS IN ARISTOTLE'S PHILOSOPHY OF SCIENCE

William Wians[1] rightly attaches great significance to the large number of mathematical examples used by Aristotle in the *Posterior Analytics,* by contrast with the *Prior Analytics* where they are quite rare. This leads one to wonder whether there are exact parallels between Aristotelian demonstration and Euclidean proof. Aristotle himself seems to assume that mathematical proofs can be given in syllogistic form, but he provides no good examples that might satisfy modern scholars like Mueller and Barnes, who find little or no fit between them. However, I am convinced that Aristotle felt that mathematical proofs could in principle be reformulated in syllogistic format (though he did not carry out this plan)[2] because he used a logical method of subtraction to explain how mathematics is possible as an exact science. For him subtraction is a logical device for identifying the primary subject of any *per se* attributes, which can then be proved to belong to such a subject in a syllogistic way.

It is clear from Aristotle's mathematical examples that he is concerned not so much with analysing the mathematical disciplines themselves as with illustrating his own theory of demonstration. For instance, he names elementary entities like the point, the line, and the unit as objects of study, while identifying number and magnitude as the genera studied by arithmetic and geometry.[3] To avoid modern misunderstandings, it is important to notice that for Aristotle the basic elements or principles of mathematics are not propositions but objects that fall naturally into different subject genera. This is the ontological basis for his famous prohibition against "crossing into another genus," e.g. trying to prove something in geometry by means of arithmetic. Thus, for instance, in *Posterior Analytics* I.9 Aristotle rejects Bryson's attempt to square the circle on the grounds that it is based on a logical fallacy, due to his failure to limit the premises to the subject genus studied by geometry. Aristotle's criticism takes for granted the discovery of incommensurability which led to a sharp distinction between arithmetic and geometry. This historical development in Greek mathematics is also relevant in I.5 where Aristotle refers to Eudoxus' general theory of proportion, remarking that the theorem about alternating proportions was once proved separately for numbers, lengths, times and solids because these were not named under a single genus. Eudoxus grouped all of these under a single comprehensive term and this somehow made possible a general theory of proportion in which certain properties can be demonstrated to belong to all of them *per se.* I will return to this historical achievement of Eudoxus later because it provides Aristotle with an important illustration for his claim that one can logically separate (by subtraction) a primary subject of *per se* attributes (thereby making demonstration possible) without ontologically separating it, as Plato is reputed to have done.

But a simple rejection of Platonism is not quite so easy for Aristotle, given that he accepts its fundamental epistemological claim that knowledge is universal (I.4-5), whereas perception is particular (I.31). Since mathematics is scientific and precise (I.13), Plato's objectivity argument implies that it must have separate objects about which it is true, given that it is not true of changing and particular sensible things. We see Aristotle squaring up to this epistemological problem at *Posterior Analytics* I.24 where he admits that if a demonstration is true then it holds true of some thing. But this seems to imply that there must be a universal object corresponding to a universal demonstration; e.g. a triangle apart from individual triangles, or a number apart from individual numbers. But at *APst.* 85b19-23 Aristotle denies the ontological implications which Platonists drew from this epistemological situation. He admits that there must exist some universal account (*logos*), which holds true of several particulars, and that this universal is imperishable. Yet he denies that this is a separately existing thing; i.e., it does not signify some individual substance but rather some quality, quantity or relation. Here Aristotle is appealing to his *Categories* (5, 2a11-b6), according to which individual substances are the basic realities, while quantities and qualities depend on substances for their existence. Thus from *Categories* 6, 4b20-5 it would appear that the objects of mathematics are either discrete or continuous quantities, so that they are attributes of substance rather than being themselves substances. However, the Platonist account cannot be wholly misguided because mathematicians treat their objects of study as if they were completely separated from sensible things.

GOING THROUGH THE PUZZLES

If one wants to understand Aristotle's problem-situation within its proper historical context, one must consider how he understood his own philosophical enterprise with respect to previous thinkers by paying particular attention to Aristotle's aporetic method, which typically begins with a review of competing opinions. Such a review is carefully constructed so as to produce an impasse which must be broken by any successful solution of the aporia. Usually the solution is already being prepared through his review of opinions, which is structured in terms of an exhaustive outline of logical possibilities. If all of the logically possible views except one have been surveyed and refuted, then the remaining logical option must be considered a likely solution. The final dialectical test which Aristotle uses for such a solution is to examine whether it "saves the phenomena" or captures the grain of truth which he finds to be present in all the reputable opinions (*endoxa*) of his predecessors.

Here, I can only sketch how this aporetic method of inquiry operates with respect to some central questions about mathematics which one finds in *Metaphysics* Beta and Kappa. The first aporia in Beta which deserves scrutiny goes as follows:

> And we must also inquire into this, (4) whether sensible substances alone should be said to exist or besides these also others, and if others also, whether such substances are of one genus or of more than one; for example, some thinkers posit the Forms and also the Mathematical Objects between the Forms and the sensible things.[4]

One can see immediately from this aporia that it is implicitly connected with the previous problem (995b10-13) about whether there is a single science dealing with all substances.[5] These questions arise as part of an extended discussion about the subject matter of his so-called science of first philosophy (or metaphysics) which Aristotle treats as if it were a science in the making. For instance, in *Metaphysics* Kappa (1059a38), he says that it is difficult to decide whether this science deals only with perceptible substances or with some other separate substances. If the latter is the case then it must deal either with the Forms or with the Mathematicals. Although Aristotle takes it to be evident that the Forms do not exist, he argues that even if one supposes them to exist, there will be a puzzle as to why there are not Forms for other things besides the objects of mathematics.

What he is raising difficulties about in *Metaphysics* Kappa is the reputedly Platonic view that the objects of mathematics constitute an intermediate class of substances between Forms and sensible things, even though no such intermediates are posited between perceptible men and the Form of Man. On the other hand, if such mathematical intermediates are not posited, then it is difficult to see what the mathematical sciences will have as objects of inquiry, since it appears that mathematics cannot be about perceptible things. This is a neat summary of the problem about the ontological status of mathematical objects, as we find it outlined both in *Metaphysics* Beta and Kappa. On the one hand, mathematics cannot be about such a class of independent substances because they do not exist, just as Platonic Forms do not exist; but, on the other hand, the mathematical sciences cannot be about sensible things which are subject to change and are perishable. So in his search for a solution to the problem Aristotle must find a middle way by discovering another mode of being for mathematical objects. For him it would be unthinkable that mathematics should not have its own proper subject matter, since this would undermine its status as a paradigmatic science of "things that can be learned" (*μαθήματα*).

The second aporia I want to consider is listed last in *Metaphysics* Beta 1, though it is closely connected with the aporia already outlined. That aporia covered mathematical objects in a general way under the question about different kinds of substance, whereas this deals more specifically with the ontological status of mathematical objects:

> Moreover, (14) are numbers and lines and figures and points substances in any sense or not, and if substances, are they separate from sensible things or are they constituents of them.[6]

When Aristotle tries to resolve this aporia in *Metaphysics* XIII, he considers precisely the same two options for mathematical objects as substances; namely as separate from sensible substances or *in* them. There he also attributes each option to some contemporary thinkers, including the Platonists, though Aristotle has changed the framework with his assumption about the primacy of sensible substances.[7]

BREAKING THE IMPASSE

Any adequate account of Aristotle's views on the ontological status of mathematical objects must take its bearings from *Metaphysics* Mu 1-3. Yet here his search for a solution to this problem takes a step beyond the aporetic strategy in Beta, where he merely reviewed the difficulties on both sides of the question. In Mu 2 he engages in elenctic argumentation by using many of the same difficulties to refute his opponents, so that in forensic terms one can say that he ceases to be an impartial judge and becomes a plaintiff in the case. This seems to be a further step in the dialectical search for truth because one should not remain bound in puzzlement forever, even though being so bound may be an essential first step towards philosophy.[8] But to break the bonds of *doxa* (typified in the review of difficulties) one needs a "hard-hitting elenchus" to clear the road into the realm of truth.[9]

Thus it is clear from Aristotle's concluding methodological remarks in Mu 1 that he regards philosophy as a shared enterprise whose ultimate goal is the extraction of truth from common opinions. As to the rationale for considering the opinions of others, he explains (1076a15-16) that one should be content if one states some things better and other things no worse. This involves some sort of elenctic test for deciding whether things are said well or badly. Indeed Aristotle espouses a rather modest ideal for philosophical inquiry, when he claims that one has done an adequate job if one formulates some theories that avoid the mistakes of previous thinkers (as exposed through a successful elenchus), while accepting those views which have survived the critical scrutiny involved in a failed elenchus. That is why one must begin every inquiry with the opinions of predecessors and pursue the truth by attempting to refute them.

For purposes of completeness, Aristotle usually classifies the opinions of predecessors in terms of the logically possible answers to a given question, and so at *Met.* 1076a32-37 he outlines the possible modes of being of mathematical objects, some of which correspond to the opinions of previous thinkers. For instance, the first logical possibility, i.e. that mathematical objects are *in* sensible things (*ἐν τοῖς αἰσθητοῖς*), corresponds to the opinion reported in *Metaphysics* Beta 2 (998a7-19).[10] By contrast, the position represented in the second logical possibility is that mathematical objects are separated from sensibles (*κεχωρισμένα τῶν αἰσθητῶν*). Although Aristotle does not identify its proponents, I think they must be 'strict' Platonists who all share the view that mathematical objects are separated from sensible things as independent substances, whether these are called Ideas or Intermediates or both. Furthermore, it corresponds exactly with one of the possibilities listed in *Metaphysics* Beta (996a12-15 & 1001b26-28) under the aporia about whether or not mathematical objects are some (kinds of) substances or not. Assuming a positive answer, the aporia lays out two possibilities for mathematical objects as substances; i.e. either separated from sensibles or belonging *in* them.

Since the first two possibilities cover the ways in which mathematical objects can exist as substances, the last two possibilities must be about alternative modes of being: (iii) either mathematical objects do not exist (ἢ *οὐκ εἰσὶν*) or (iv) they exist in some other way (ἢ *ἄλλον τρόπον εἰσὶν*). The third possibility is included only for the sake of logical completeness, as Aristotle does not consider it further. This apparent oversight can be explained away by reference to the Platonic argument "from the sciences," whose fundamental assumption is that any genuine science must have a real or existent object.[11] Since Aristotle shares that assumption, he would probably find it unthinkable that the objects of mathematics should not exist at all because that would leave these paradigmatic sciences without foundations.

So, if the first two possibilities are to be denied and the third be ruled out, the remaining option takes on a new importance. As stated, this is the possibility that mathematical objects exist in some other manner. Obviously, it must be some mode of being which lies between complete non-being and being in the primary sense as substance. However, Pseudo-Alexander[12] is premature in describing this mode of being as "abstract" (*ἐξ ἀφαιρέσεως*), since Aristotle's own account emerges from the dialectical inquiry rather than being a presupposition for it.

It is from this dialectical perspective that we should view any argument which serves as a refutation in Mu 2 and which is used again in Mu 3 to support Aristotle's own positive solution, since it illustrates perfectly the complex role which difficulties play in his procedure. On the one hand, they provide the material for refuting an opponent's view while, on the other hand, they also belong among the phenomena to be 'saved' by any solution that emerges from the process of refutation. In this case, Aristotle bases his objection against the Platonists on the development of a general theory of proportion by mathematicians within the Academy:

> Again, some mathematical propositions are universally expressed by mathematicians in such a way that the objects signified are distinct from these mathematical substances. Accordingly, there will be other substances which are separate, which lie between the Ideas and the Intermediates,

> and which are neither specific numbers nor points nor specific magnitudes nor time. If this is impossible, it is clear that the others, too, cannot exist separate from the sensible substances.[13]

Although Aristotle does not specify the referent, the passage indicates it is some kind of universal (*καθόλου*) theory in mathematics whose range is not limited to any particular quantity, such as the general theory of proportion in Book V of Euclid's *Elements*.[14]

In order to illustrate Aristotle's point here, Ps.-Alexander (729.21 ff) supplies an example from this theory and another from the general axioms of equality, while Syrianus (89.30 ff.) also cites the same two examples. Similarly, modern commentators treat Eudoxus' theory of proportion as the best example of such a universal mathematics.[15] From this historical perspective, one can now see the power of Aristotle's objection when it is directed against the Platonists, especially those who accepted the general theory of proportion. Given that this theory is not specifically about numbers or points or lines or any of the other kinds of continuous magnitude, which the Platonists considered to be separate substances, they are faced with the following difficulty. One implication (1077a10-11) of their position, when applied to the general theory of proportion, is that there must be some other substance which is separated from and between (*μεταξὺ*) Ideas and Intermediates. Furthermore, (to compound the difficulty) such a substance cannot be either a number or a point or a magnitude or time. If this result is impossible, as appears to be the case, then it is also impossible for these other mathematical objects to exist apart from sensible things. The whole objection depends on the assumption that the separation of mathematical objects involves treating them as independent substances.

In the final argument of Mu 2, Aristotle identifies the nub of his dispute with the Platonists about mathematical objects:

> Let it be granted that they are prior in formula to the body. But it is not always the case that what is prior in formula is also prior in substance. For A is prior in substance to B if A surpasses B in existing separately, but A is prior in formula to B if the formula of A is a part of the formula of B; and the two priorities do not belong to the same thing together. For if attributes, as for example a motion of some kind or whiteness, do not exist apart from substances, whiteness is prior in formula to the white man but not prior in substance; for whiteness cannot exist separately but exists always in the composite. By "the composite," here, I mean the white man. So, it is evident that neither is the thing abstracted prior, nor is what results by addition posterior; for it is by addition of whiteness that we speak of a white man.[16]

The initial concessive *μὲν* here shows that Aristotle is prepared to accept that mathematical objects are prior in definition (*τῷ λόγῳ πρότερα*) to sensible bodies, but he minimizes the concession by saying that not all things which are prior in definition are also prior in substance (*τῇ οὐσίᾳ πρότερα*). He supports this distinction by citing different criteria for the two types of priority. Some thing A is prior in substance to something else B if A surpasses B in existing separately, whereas A is prior in definition to B if the definition of A is part of the definition of B. Aristotle warns that the two types of priority do not always belong to the same thing.[17]

Despite the clear logical basis for Aristotle's argument, one might still ask how it is an objection to the Platonist claims about the ontological status of mathematical objects. Given the whole topic of the treatise, it is rather curious that he chooses a quality like whiteness rather than some quantity, in order to make his point about the non-coincidence of two kinds of priority. According to his own categorial framework, however, both quantities and qualities are accidents of primary substance and so can be defined separately from it. Thus the point of Aristotle's example is to suggest that the Platonists have been misled by this logical possibility. The fact that whiteness can be defined independently of sensible substances does not mean that there is some Whiteness Itself apart from sensible things, as the Platonists thought; cf. *Phy.* 193b35ff. Although mathematical quantities are more separable from sensible things than qualities, one cannot infer that they are independent substances from the fact that their definitions do not presuppose any sensible subjects to which they belong *per se.*

This is the general thrust of Aristotle's rather strange conclusion (1077b9-11) in the present passage to the effect that "the result of subtraction" (τὸ ἐξ ἀφαιρέσεως), is not prior nor is "the result of addition" (τὸ ἐκ προσθέσεως) posterior. The terminology of 'abstraction' is introduced quite suddenly, and the context provides little guidance as to how it should be interpreted, except for an explicit contrast with some process called "addition." Fortunately, Aristotle does give us a clue as to what he means by 'addition' when he says that it is as a result of adding to whiteness that the white man is spoken of.[18] From the previous passage we may assume that he is here referring to the addition of a subject (i.e. "man") that is not the primary subject to which the quality of whiteness belongs *per se.* Conversely, "abstraction" would be the process of taking away that subject and defining white separately. This is consistent with Aristotle's denial of priority to "the result of subtraction," since he had previously argued that "the white" is not prior in substance to "the white man" even though it may be prior in formula.

In fact, it is quite clear that priority in substance is being denied to the so-called "results of subtraction." This may have led some ancient Greek commentators to the conclusion that Aristotle is here referring specifically to mathematical objects.[19] Yet they give no adequate explanation of how mathematical objects could be intelligibly referred to as "the results of abstraction" or of what implications this terminology has for their ontological status. This is a lacuna even in modern Aristotelian scholarship, which needs to be filled by explaining such terminology and by showing how it describes the logical situation of mathematical objects. Such an analysis must also explain the peculiar fact that the terminology of "abstraction" is not used by Aristotle in Mu 3 for his positive account of the mode of being of mathematical objects.[20]

PROVIDING SOLUTIONS TO THE *APORIAI*

Having refuted the views of others, Aristotle's next task is to provide an alternative account of mathematical objects which will escape the difficulties raised. If his solution manages to do this, while also saving the most authoritative phenomena, then it will be a successful resolution of the problem according to his methodological criteria. Among these phenomena we expect to find the reputable opinions (*endoxa*) of mathematicians who are the 'wise' in this case. Thus it is not surprising that Eudoxus' general theory of proportion is made the starting-point for Aristotle's own proposed solution:

> Now, just as certain universal propositions in mathematics, which are about things not existing apart from magnitudes and numbers, are indeed about numbers and magnitudes but not qua such as having a magnitude or being divisible, clearly, so there may be propositions and demonstrations about sensible magnitudes, not qua sensible but qua being of such-and-such a kind.[21]

Here Aristotle appeals to the fact that mathematicians use general axioms and propositions about quantity as such without positing other objects besides magnitudes and numbers. Structurally, the argument draws a parallel between the fact that there are such general propositions and the possibility that other statements and proofs can be made about sensible magnitudes. The first part of the parallel assumes as established that the propositions of general mathematics are not about separated things apart from magnitudes and numbers. Yet, while a proposition from the general theory of proportion is about magnitudes and numbers, it is not about them in so far as [ᾗ] these things have continuity or are discrete.

Therefore, starting from the general theory of proportion, Aristotle draws a parallel which is crucial for his alternative account of all the sciences as being about sensible things. He claims (*Met.* 1077b20-22) that, in a similar way, there can be propositions and proofs about sensible magnitudes, not insofar as they are sensible but insofar as they are such-and-such [*ἀλλ' ᾗ τοιαδί*]. What he appears to mean by this claim is that one can select some definite quality [*τοιαδί*] of sensible magnitudes and construct demonstrations with respect to it as subject, while excluding the sensible aspects from consideration. Thus he makes the following loose analogy: just as there are propositions about quantity as such, which leave out of account whether the quantity is continuous or discrete; so also there are propositions about sensible magnitudes which do not consider them as sensible but only as magnitudes.[22]

Let us now consider how Aristotle's use of Eudoxus' theory has advanced his alternative account of mathematical objects. The argument based on the theory of proportion draws the following logical parallel: just as it is possible to have a science about numbers and magnitudes in so far as they are quantities, without the ontological separation of some entity called "quantity"; so also one can have a science of sensible magnitudes in so far as they are such and such [*ᾗ τοιαδί*]. Perhaps Aristotle is being deliberately vague here so as to make the point that the "*qua*" locution can pick out any aspect of sensible magnitudes and bring it under the subject matter of a particular science. It also establishes the possibility of demonstrative knowledge of that unseparated aspect because the "*qua*" locution indexes the primary subject of whatever attributes are proved to belong to something qua such-and-such.[23]

Now it is upon this logical basis that Aristotle continues to build his argument as follows:

> For just as there are many propositions concerning sensible things but only qua moving, without reference to the whatness of each of these and the attributes that follow from it – and it is not necessary because of this that there should exist either a moving of a sort which is separate from the sensible thing or is some definite nature in the sensible thing – so also there will be propositions and sciences about things in motion, not *qua* in motion but only *qua* bodies, or only *qua* planes, or *qua* lengths, or *qua* divisible, or *qua* indivisible with position, or just *qua* indivisible.[24]

As in the previous argument, the general structure of this argument is that of an explicit parallel which is drawn between an actual and a possible situation. Here Aristotle starts from the existence of many statements about things only in so far as they are changing [*ᾗ κινούμενα μόνον*], quite apart from the particular essence of such things or their accidents.

It is clear that what he is proposing as a basis for the truth and objectivity of any science is the possibility of logically separating its subject-matter from the complex appearances of sensible things. For instance, he emphasizes that we are able to make true statements about sensible things *qua* moving, while leaving out of account the essence of these things along with all other accidental attributes. Obviously, such a leaving out is logical because the essence of anything is ontologically inseparable from it and could not be ignored, for instance, if we were considering something under its species description. It is important to notice, however, that Aristotle mentions the possibility of leaving the essence out of account through this logical technique of subtraction. If this were not possible then there would be only one science of sensible things; e.g. a science of natural kinds. But he clearly rules out this possibility at the end of the above passage when he draws the second part of his parallel: just as there are propositions about sensible things qua moving, so also there can be propositions and sciences about moving things, not qua moving but qua bodies only [ᾗ *σώματα μόνον*]. In other words, just as one can select some aspect of sensible things as a primary subject for attributes related to motion, so also one can select the bodily aspect of moving things as a subject to which some attributes belong primarily and universally.

Despite the ambiguity of the word *σώματα*, it seems very likely that Aristotle has in mind the solids [*στερέα*] whose *per se* attributes are studied by the science of stereometry. The selection of solids as the primary subject of such attributes is indicated by the "*qua*" locution and is achieved through subtraction. Indeed, the passage goes on to list a series of such subtractions which itself seems to have an inherent order. First, one considers moving or changing things, not *qua* moving but only *qua* solids. This step involves the logical subtraction of the sensible and changing aspects of things, together with the *per se* attributes that belong primarily to this aspect; e.g. sensible contraries like hot/cold, light/heavy, wet/dry. The analogous step in *Posterior Analytics* I.4 (74a33-b4) is the subtraction of "bronze" from the complex subject "bronze isosceles triangle," thereby eliminating certain sensible attributes. Such a logical step makes possible the isolation of the solid as a primary subject for the attributes which stereometry will demonstrate as belonging to it *per se.*

The method of subtraction can be used again in a logical way to "strip off" [ἀ*φαιρ*εῖ*ν*] the third dimension and thereby eliminate its *per se* attributes; cf. *Met.* Z 3, 1029a10 ff. & K 3, 1061a28 ff. This is presumably what Aristotle has in mind at Mu 3 when he says that there can be a science of sensible things *qua* planes [ᾗ ἐ*πίπεδα*]; i.e. plane geometry. Similarly, the second dimension can be logically removed so as to make possible the study of sensible things *qua* lengths [ᾗ *μήκη*]. The method of subtraction allows one to identify certain attributes as belonging universally to the line as a primary subject; e.g. straight and curved belong to bodies in so far as they contain lines. Therefore, strictly speaking, it is only *qua* line that a sensible thing can be said to be either straight or curved. Although Aristotle does not mention Protagoras within this context, one can now see how one might defuse his well-known objection that mathematical definitions (e.g. for the tangent of a circle and a line) are not true of sensible things. When Protagoras objects that a sensible circle and ruler do not meet at a point, he is wrongly assuming that this property belongs to the contact of the circle and the line in so far as they are sensible. In general, this mistake is being made by anyone who appeals to some empirical fact about a sensible diagram in order to refute a geometrical claim.

In terms of his whole project in *Metaphysics* Mu 1-3, however, we would expect Aristotle to specify an alternative mode of being for mathematical entities which conforms with the actual practice of mathematicians, as he does in the following passage:

> A thing can best be investigated if each attribute which is not separate from the thing is laid down as separate, and this is what the arithmetician and the geometrician do. Thus, a man *qua* a man

> is one and indivisible. The arithmetician lays down this: to be one is to be indivisible, and then he investigates the attributes which belong to a man *qua* indivisible. On the other hand, the geometrician investigates a man neither *qua* a man nor *qua* indivisible, but *qua* a solid. For it is clear that the attributes which would have belonged to him even if somehow he were not indivisible can still belong to him if he is indivisible. Because of this fact, geometers speak rightly, and what they discuss are beings, and these are beings; for "being" may be used in two senses, as actuality and as matter.[25]

What Aristotle here proposes as a solution, i.e. that mathematical objects exist "as matter" [ὑλικῶς], has itself prompted many different interpretations.[26] Instead of rehearsing these views, I will follow the hermeneutical maxim that Aristotle's brief and ambiguous solution must be interpreted in terms the whole aporetic inquiry.[27]

The above passage begins with a methodological recommendation for the other sciences based on the procedure of mathematicians. I take the word *οὕτω* to refer back to that procedure, which is then redescribed in a conditional clause as follows: "if one posits as separate what (in reality) is not separated . . ."[28] This clause contains a clear contrast between the logical and ontological implications of the positing activity of the arithmetician and the geometer. While their subject-matter may be treated as logically separate, Aristotle insists that it is not separated in reality. Therefore he recommends this procedure for each of the other sciences because it promotes greater accuracy without leading to error.

The most obscure part of this passage is the description of how the arithmetician con siders a man as one indivisible thing, while the geometer treats him as a solid. One may be tempted to object that the mathematician does not deal with man at all, whether as unit or as solid, but that would be to miss the whole point of his argument.[29] For Aristotle does not want to claim that mathematics is about mankind, though he does wish to establish that these sciences can be viewed as dealing with sensible things under highly specific aspects. Obviously, he is concerned with the truth of mathematics which, according to his cor respondence theory, depends on the existence of real entities. For instance, the statement about the arithmetician begins with an explicit comparison between what is posited by him and what is actually the case. On the one hand, Aristotle says, a man *qua* man is one and indivisible [*ἓν μὲν ... καὶ ἀδιαίρετον*] while, on the other hand, the arithmetician posits the unit as indivisible [*ὁ δ᾽ ἔθετο ἓν ἀδιαίρετον*] and then considers whether any attributes belong to the man *qua* indivisible. The point implicit in the Greek construction seems to be that the arithmetician has not assumed any falsehood, despite the fact that he posits the unit as if it were independent of the sensible world. Aristotle's use of the aorist here, combined with a temporal index word [εἶτ᾽], suggests that the arithmetician simply goes ahead and posits an indivisible unit without reflecting on his ontological assumptions, and this conforms quite well with what Aristotle says elsewhere[30] about the practice of mathematicians. In fact, he does not think it is any part of their business to investigate foundational questions.[31] As a philosopher, however, Aristotle must ground the mathematical sciences in the reality of the sensible world, especially since he has undermined the foundations which the Platonists gave them in the supersensible realm.

In the present passage, therefore, he tries to establish that these sciences are true of sensible things under a certain description. For instance, one can count men without falling into error because a man *qua* man conforms to the definition of a unit which is posited by the arithmetician. By contrast, if one tried to count the same things *qua* colored, the possibility of error and confusion is greater. In modern jargon, one might formulate the difference between "man" and "color" as follows: whereas the former is a sortal term that divides its reference cleanly, the latter is a mass term that does not.[32] There is also some basis for a corresponding distinction in Aristotle's work where he recognises that only certain concepts provide us with a measure for counting a collection of things; cf. *Met.* 1014a26-31, 1088a4-11. Here he specifies very carefully the aspect under which an arithmetician might consider a sensible thing such as a man. Even though a man is one and indivisible in so far as he is a man (i.e. under the species description), the arithmetician is not interested in him as such; otherwise he would be engaged in some kind of biology. Indeed, the mathematician only deals with a man in so far as he is an indivisible unit and so far as some numerical attributes belong to him under that description.

All ancient varieties of Platonism are being resisted by Aristotle as he struggles to find a plausible way of connecting the science of geometry with sensible things. This is why he uses a counter-factual conditional to talk about what could belong to a man if he were not indivisible, and so it is only through the 'qua' locution that he can establish the logical possibility of talking about a man insofar as he is a solid [ᾗ *στερεόν*]. When this aspect has been isolated as a primary subject, it is possible to claim without contradiction that a man has certain *per se* attributes which are directly opposed to those which belong to a man *qua* unit. In addition to the logical situation, however, the mode of being of this aspect must be clarified before one can be assured of the truth of geometry as a science concerned with sensible things. This appears to be what Aristotle has in mind when he insists that geometers speak correctly [*ὀρθῶς*] and that they are speaking about "beings" [*ὄντα*] which really do exist. In support of this claim, he appeals to two general senses in which "being" is used; namely, being in the sense of actuality [*ἐντελεχείᾳ*] and being in a material sense [*ὑλικῶς*]. Given the familiar look of this distinction, it is natural to think that *ὑλικῶς* must stand for potential being, but yet we must wonder about Aristotle's reasons for choosing this word rather than *δύναμις*. To grasp his meaning, however, we should confine ourselves to asking how the conclusion should be understood within the context of the whole argument in Mu 1-3, especially in view of the linguistic hint that mathematical objects may have a mode of being analogous to that of matter rather than to that of substantial form. The simplest way to interpret this hint is that mathematical objects have a dependent mode of being by contrast with the independence that is characteristic of substances. But, in order to save the phenomena, this must also provide a solution that satisfactorily resolves the difficulties raised in *Metaphysics* Beta.

Firstly, it clearly avoids all the difficulties arising from treating mathematical objects as independent substances either *in* sensible things or separate from them, since Aristotle denies them the mode of being of substantial forms. Furthermore, given that mathematical bodies are not substantial, they will not be competing for the same place with physical bodies, since they are potentially but not actually in sensible things. Just as the statue of Hermes is potentially in the marble block before it has been sculpted, so the geometrical lines, planes and solids are potentially in sensible objects before they have been separated out by the method of subtraction. But this parallel also tends to suggest that the mathematician is like a craftsman who actively shapes the matter which would remain merely potential without his agency, and it is unclear whether Aristotle is committed to such an implication. In *Metaphysics* Beta he does talk about the "generation" of geometrical divisions but that is an instantaneous rather than temporal process, so that it is quite different from any kind of physical or artistic generation. However, there may be some parallels with the activity of the intellect in grasping mathematical objects which are paradigmatic "things to be learned."

CONCLUSION

Returning to my hermeneutical point of departure, I want to reconsider whether Aristotle's philosophy of mathematics can be expressed in any of the standard modern views such as platonism, logicism, formalism, intuitionism, or quasi-empiricism. Given his rejection of ancient Platonism in mathematics, it would seem difficult to treat him as a platonist even though he does accept that mathematical objects are real entities independent of the human mind. Yet this would make him a realist at least, and perhaps even a platonist like Frege. But a better case might be made for treating him as a logicist, given the logical basis for his theory of subtraction that grounds his account of the mathematical sciences.[33] However, this does not seem to fit either because Aristotle regards logic as preparatory for the sciences, whereas mathematics is one of the theoretical sciences. Unlike Frege and Russell, he makes no attempt to reduce mathematics to logic and his defense of the principle of contradiction in *Metaphysics* IV relies more on ontology than on logic. In fact, given the explicit parallels which Aristotle draws between mathematics and physics, one might try to classify him as a quasi-empiricist like Lakatos who insists that mathematics has many experiential and *a posteriori* elements just like physics. But again Aristotle never draws a clear distinction between *a priori* and *a posteriori* propositions, and his model of mathematics as a demonstrative science does not fit very well with the quasi-empiricism of Lakatos and his more radical followers.

On the other hand, given Aristotle's views on the potential infinite, it would appear that he should be classified as an intuitionist like Brouwer and Heyting. Yet, as Lear rightly points out,[34] we must be wary of the apparent similarity between these ancient and modern views. While Aristotle makes the potential infinite dependent on the nature of magnitude itself, modern intuitionists make it dependent on the existence of a finite process carried out by the creative mathematician. This difference in emphasis nicely illustrates the post-Cartesian shift in perspective from an object-centered to a subject-centered epistemology. Indeed, from this post-Cartesian perspective, we can better understand the difficulty of classifying Aristotle's philosophy of mathematics in terms of any contemporary view. The sceptical gap that Descartes opened up between knower and object known led modern philosophers to focus on questions about subjectivity and objectivity in science, rather than on questions about truth as simple correspondence between the object known and the knower. It is precisely because of sceptical doubts about the human mind's access to reality that the distinction between *a priori* and *a posteriori* propositions became relevant. Within this modern problem-situation, British empiricists such as Locke and Hume tried to combat scepticism by appealing to abstraction as an epistemological process by means of which the human mind can begin from sense experience and reach universal knowledge. Such an appeal to a traditional Aristotelian view seemed to be legitimated by ancient and medieval commentators on Aristotle who described his epistemology in terms of abstraction. However, Frege's critique of abstractionism as a psychological theory made it appear unsustainable, so that Aristotle's epistemology lost the legitimacy which it seemed to have for British empiricists. Yet, if I am correct about Aristotle not being an epistemological abstractionist, one might still treat Aristotle as a logical realist like Frege himself. In any case, whatever modern parallels one draws with Aristotle's position, it should be clear that all of them will tend to be misleading unless one pays close attention to the different problem-situations involved.

Boston College / NUI Maynooth, Ireland

NOTES

[1] See W. Wians, "Scientific Examples in the Posterior Analytics" in Wians, ed., *Aristotle's Philosophical Development: Problems and Prospects* (Lanham: University of America Press, 1996), 131-150.

[2] But see *Posterior Analytics* II.11 (94a28-31) where Aristotle puts a Euclidean proposition (*Elements* III.31) into syllogistic format. *Aristotle's Prior and Posterior Analytics,* ed. W.D. Ross (Oxford: Oxford University Press, 1949). [*APst.*]

[3] Cf. *APst.* 71a15, 72a21, 75b2-5, 84a11-26, 88b26, 90b33, 93b24.

[4] *Met.* 995b13-18: *Aristotle's Metaphysics,* trans. H.G. Apostle (Grinnell: Peripatetic Press, 1979). [*Met.*] The parallel aporia in *Met.* XI. 1 (1059a37 ff.) goes as follows: "In general, there is this problem, whether the science we now seek is concerned at all with sensible substances or not, but rather with some other substances. If with others, it would be either with the Forms or with Mathematical Objects."

[5] Cf. Alexander, *in Metaph.* 175.14-176.16. Syrianus (*in Metaph.* 2.15 ff.) goes one better by combining three *aporiai* together, though he quotes and discusses each one separately.

[6] *Met.* 996a12-15.

[7] Perhaps it is by way of reaction against this assumption that the Neoplatonic commentator, Syrianus (*in Metaph.* 12.25 ff.) adopts the strategy of simply asserting the Platonic order of priorities, beginning with the Forms of mathematical objects and concluding with their appearance in sensible things.

[8] Thus Aristotle's methodological attitude differs fundamentally from that of the ancient Sceptics who used the aporetic method as an end in itself within their philosophical inquiries.

[9] In terms of his method, therefore, Aristotle owes something to "father Parmenides" but his greatest methodological debt is to Plato's *Parmenides* and its deliberately constructed antinomies. At *Parm.* 136c5 Parmenides recommends the gymnastic exercise of constructing antinomies as a way of seeing the truth more completely (*τελέως*) and better (*κυρίως*). See M. Schofield, "The Antinomies of Plato's Parmenides," *Classical Quarterly* 27 (1977): 140-58.

[10] In order to signpost this view as it is represented by Aristotle, I adopt the convention of italicizing the 'in' as follows: ". . . mathematical objects *in* sensible things."

[11] In addition, Aristotle connects the argument "from the sciences" with the Parmenidean dictum that it is impossible to think or inquire about not-being; cf. *Cael.* III.1, 298b17-25.

[12] Cf. *In Metaph.* 725.4.

[13] *Met.* 1077a9-14.

[14] Cf. T.L. Heath, *The Thirteen Books of Euclid's Elements.* 3 vols (Cambridge: Cambridge University Press, 1925), ii, 112 ff.

[15] Cf. Heath, 138; W.R. Knorr, *The Evolution of the Euclidean Elements (*Dordrecht-Boston: Reidel, 1975); D.R. Lachterman, *The Ethics of Geometry. A Genealogy of Modernity* (New York & London: Routledge, 1989), Ch. 2.

[16] *Met.* 1077a36-b11.

[17] Cf. Cleary, *Aristotle on the Many Senses of Priority* (Carbondale: Southern Illinois University Press, 1988) for discussion of the different senses of priority in Aristotle.

[18] ἐκ προσθέσεως γὰρ τῷ λευκῷ ὁ λευκὸς ἄνθρωπος λέγεται – *Met.* 1077b11.

[19] Cf. Ps.-Alexander *In Metaph.* 733.23-24 & Syrianus, *In Metaph.* 93.22 ff.

[20] D. D. Moukanos, *Ontologie der 'Mathematika' in der Metaphysik des Aristoteles* (Athens: Potamitis Press, 1981), 24 ff., claims that the conclusion of Mu 2-3 is that mathematics is about abstract objects, which exist through the separating reflection of mathematicians, but he fails to explain why the terminology of abstraction is conspicuously absent from Mu 3. For my explanation see Cleary, "On the Terminology of

Abstraction in Aristotle," *Phronesis* 30 (1985): 13-47.

21 *Met.* 1077b17-22.

22 Syrianus (*In Metaph.* 95.13-17) expresses some surprise at what he sees as Aristotle's attempt to find a parallel in ontological status between universals and mathematical objects, since the former are logical entities belonging in the soul, whereas the latter are in sensibles and are also mental abstractions [*ἐπινοίαις ἀφαιρούσαις*] from sensibles. But such remarks cannot be taken to represent Aristotle's views accurately, and they may even suggest that abstractionism was a product of commentators like Alexander, who proposed it as the official Aristotelian doctrine that was later opposed by Syrianus.

23 In his logical analysis of what he calls Aristotle's theory of reduplication, Allan Bäck, *On Reduplication. Logical Theories of Qualification* (Leiden: Brill, 1996) points out that a *qua* proposition is actually a condensed demonstrative syllogism in which the *qua* term functions as a middle term and as a cause; e.g. an isosceles triangle has this property because it is a triangle. He also argues that the qua phrase is attached to the predicate and does not change the reference of the subject term, which he takes to be a particular existent like this bronze triangle. He has objected (in personal communication) that my approach of making qua propositions fix our attention on the primary subject has the consequence of changing the reference of the subject term to some kind of Platonic entities about which it would be difficult to verify any knowledge claims. But I respond that the distinction between natural and logical priority in Aristotle separates the *de dicto* question of the primary logical subject from the *de re* question about the basic subject as substance.

24 *Met.* 1077b22-30.

25 *Met.* 1078a21-31.

26 F.A.J. de Haas, "Geometrical Objects in Aristotle," (unpublished mss.) finds two major types of interpretations within the range given by scholars like I. Mueller, "Aristotle and the Quadrature of the Circle" in N. Kretzmann, ed., *Infinity and Continuity in Ancient and Medieval Thought* (Ithaca: Cornell University Press, 1982), 146-64; J. Lear, "Aristotelian Infinity," *Proceedings of the Aristotelian Society* 1979-80: 188-210; J. Barnes, "Aristotelian Arithmetic", *Revue de Philosophie Ancienne* 3 (1985): 97-133; M. Mignucci, "Aristotle's Arithmetic", in Graeser ed., *Mathematics and Metaphysics in Aristotle*, 175-211; Annas, "Die Gegenstande der Mathematik bei Aristoteles," Graeser, *Mathematics and Metaphysics in Aristotle*, 131-47; Modrak, "Aristotle on the Difference between Mathematics and Physics and First Philosophy" in Penner & Kraut, eds., *Nature, Knowledge, and Virtue* (Edmonton: Academic Printing, 1989), 121-39; E. Hussey, "Aristotle on Mathematical Objects" in Mueller, ed., *Peri Ton Mathematon* (*Apeiron* 24.4) (Edmonton: Academic Printing, 1991), 105-33.

27 Hussey (cited above) recognizes that Aristotle's discussion in *Metaphysics* Mu 3 is incomplete on its own, but he fails to see the broader aporetic context within which one should understand the solutions given there. Although Barnes and Annas (both cited above) insist that the solution must be seen exclusively in terms of the inquiry at Mu 1-3, yet that context is surely too narrow.

28 εἴ *τις τὸ μὴ κεχωρισμένον θείη χωρίσας* – *Met.* 1078a21-22.

29 If one accepts Frege's analysis of number as a second-order property, one might still object that Aristotle is simply wrong to think of it as a first-order property of sensible things. But that would be a different objection from the one that I describe as missing the point.

30 Cf. *APst.* 76a31-36, 76b3-11, 92b15-16, 93b21-28.

31 Cf. *Met.* 1025b3-18, 1059b14-21, *Phy.* 184b25-185a5.

32 Cf. P.F. Strawson, *Individuals* (London: Methuen, 1959), 167 ff.

33 In fact, R. Netz, *The Shaping of Deduction in Greek Mathematics* (Cambridge: Cambridge University Press, 1999), 214 claims that Aristotle should be regarded as a logicist in the modern sense but I think that Netz fails to take into account the different problem-situations that prevailed in the widely separated historical eras.

34 J. Lear, *Aristotle: The Desire to Understand* (Cambridge: Cambridge University Press, 1988), 68n34.